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COMMENT

Comments on ferroelectric ice model on a triangular lattice†

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Abstract. It is shown that the free energy per vertex for a ferroelectric ice model on a triangular lattice can be expressed in terms of elementary functions. An exact upper bound for the residual entropy of ice on a Kagomé lattice is obtained.

Kelland (1974a) solved exactly a ferroelectric ice model on a triangular lattice. The free energy per vertex $\mathscr{F}$ is given by the principal value integral (we follow the notation of Kelland)

$$-\beta\mathscr{F} = \mathrm{P}\int_{-\infty}^{\infty} \frac{16\sinh(\pi-\phi)m\cosh^3(\pi-\phi)m}{m(\mathrm{e}^{2\phi m}+1)(1-\mathrm{e}^{-2\pi m})}\,\mathrm{d}m \tag{1}$$

where

$$2\cos\phi = -\sqrt{(1+u^{-1})} \qquad \tfrac{2}{3}\pi < \phi < \pi.$$

The integral (1) was later integrated by Kelland to give (Kelland 1974b, equations (57) and (58))

$$-\beta\mathscr{F} = 2\int_{-\infty}^{\infty} \frac{\sinh^2(\pi-\phi)m\cosh^3(\pi-\phi)m}{m\cosh(\phi m)\sinh(\pi m)}\,\mathrm{d}m \tag{2}$$

$$= -\sum_{m=1}^{\infty}\frac{2}{m}\sin^2(2m\phi) + \sum_{m=1}^{\infty}\frac{4}{2m-1}\sin^2[(2m-1)\pi^2/\phi]. \tag{3}$$

It is interesting to note that Kelland's expression (3) can be evaluated in terms of elementary functions as follows:

$$-\beta\mathscr{F} = \lim_{N\to\infty}\left(-\sum_{m=1}^{N}\frac{2}{m}\sin^2(2m\phi) + \sum_{m=1,2,\ldots}\frac{4}{2m-1}\sin^2[(2m-1)\pi^2/\phi]\right)$$

$$2m-1 < 2\phi N/\pi$$

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and so

$$-\beta\mathscr{F}=\lim_{N\to\infty}\left(-\sum_{m=1}^{N}\frac{1}{m}+\sum_{m<\phi N/\pi}\frac{2}{2m-1}\right)$$
$$+\sum_{m=1}^{\infty}\frac{\cos(4\phi m)}{m}-\sum_{m=1}^{\infty}\frac{2}{2m-1}\cos\left(\frac{(2m-1)2\pi^2}{\phi}\right)$$
$$=\ln\left(\left|\frac{2\phi\tan(\pi^2/\phi)}{\pi\sin(2\phi)}\right|\right). \tag{4}$$

In particular, the case $u=1$ corresponds to the triangular ice model of Baxter (1969) and $-\beta\mathscr{F}=\ln W$ where $W=3\sqrt{3}/2=2{\cdot}598\ldots$.

Recently it has been shown by Lin (1976) that the ice model on a Kagomé lattice (Lin and Tang 1976) is equivalent to a special case of the 20-vertex model on a triangular lattice such that (in the notation of Kelland)

$$u_1=u_2=u_3=u_5=u_8=u_9=1 \qquad u_4=u_6=u_7=u_{10}=2. \tag{5}$$

Since the soluble condition (Kelland 1974b, equation (19)) is not satisfied, the residual entropy of ice on a Kagomé lattice cannot be obtained by the method of Bethe *ansatz*. Instead we have an exact upper bound for the ice model on a Kagomé lattice:

$$W(\text{Kagomé lattice})<(e^{-\beta\mathscr{F}/3})_{u=2}=1{\cdot}708\ldots. \tag{6}$$

This upper bound is an improvement on that obtained by Lin (1976) of $3^{1/2}=1{\cdot}732\ldots$.

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